

Existence of positive solutions for an  $(n + 2)$ -order nonlinear BVPFu-Hsiang Wong<sup>a,\*</sup>, Sheng-Ping Wang<sup>b</sup>, Shiueh-Ling Yu<sup>c</sup>, Wei-Cheng Lian<sup>d</sup><sup>a</sup> Department of Mathematics, National Taipei University of Education, 134, Ho-Ping E. Rd, Sec2, Taipei 10659, Taiwan, ROC<sup>b</sup> Department of Mathematics, TamKang University, Taipei, Taiwan<sup>c</sup> Holistic Education Center, St. John's and St. Mary's Institute of Technology, Tamsui, Taipei, Taiwan, ROC<sup>d</sup> National Kaohsiung Marine University, No. 142, Hai Chuan Road, Nan-Tzu Dist, Kaohsiung, Taiwan, ROC

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## ABSTRACT

Under some suitable assumptions, we shall show that the  $(n + 2)$ -order nonlinear boundary value problem:

$$(BVP) \begin{cases} (E) [\phi_p(u^{(n)}(t))]'' = f(t, u(t), u^{(1)}(t), \dots, u^{(n+1)}(t)) \\ \begin{cases} u^{(i)}(0) = 0, i = 0, 1, 2, \dots, n-3, \\ u^{(n-2)}(0) = \xi u^{(n-2)}(1), \\ u^{(n-1)}(1) = \eta u^{(n-1)}(0), \\ u^{(n)}(0) = \mu u^{(n)}(\delta), \\ u^{(n)}(1) = \nu u^{(n)}(\delta), \end{cases} \end{cases} (BC)$$

has at least three solutions in  $C^n[0, 1]$ ,  $n \geq 2$ .

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## 1. Introduction

In the last thirty years, a great deal of work has been done to study the positive solutions of the two-point boundary value problem for differential equations which are used to describe a number of physical, biological and chemical phenomena. For additional background information and results, we refer the reader to the monograph by Agarwal and Wong [1–5], Agarwal, O'Regan and Wong [6] as well as recent contributions [7–11,14–19].

Boundary value problems for higher-order differential equations can arise, especially for fourth-order equations. Recently, the three-point or multi-point boundary value problem of the differential equations was presented and studied by many authors; see [12,13,16,17].

In this paper, we attempt to establish some existence theorems of positive solutions for the following  $(n + 2)$ -order nonlinear boundary value problem:

$$(BVP) \begin{cases} (E) [\phi_p(u^{(n)}(t))]'' = f(t, u(t), u^{(1)}(t), \dots, u^{(n+1)}(t)) \\ \begin{cases} u^{(i)}(0) = 0, i = 0, 1, 2, \dots, n-3, \\ u^{(n-2)}(0) = \xi u^{(n-2)}(1), \\ u^{(n-1)}(1) = \eta u^{(n-1)}(0), \\ u^{(n)}(0) = \mu u^{(n)}(\delta), \\ u^{(n)}(1) = \nu u^{(n)}(\delta), \end{cases} \end{cases} (BC)$$

where  $f : (0, 1) \times \mathbb{R}^{n+2} \rightarrow [0, +\infty)$  is a continuous function;  $\mu, \nu \geq 0, \xi \neq 1, \eta \neq 1, 0 < \delta < 1, n \geq 2$  and  $\phi_p(z) = |z|^{p-2}z$  for  $p > 1$ .

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In 2006, Ma and Ge [18] studied this topic for the boundary value problem:

$$(BVP_0) \begin{cases} (E_0) [\phi_p(u''(t))]'' = a(t)f(u(t)) \\ u(0) = \xi u(1), \\ (BC_0) \begin{cases} u'(1) = \eta u'(0) \\ u''(0) = \mu u''(\delta) \\ u''(1) = \nu u''(\delta). \end{cases} \end{cases}$$

They applied a fixed-point theorem to establish the existence of at least three positive solutions of  $(BVP_0)$ . Now, we consider the more general case (BVP) and hope to obtain some extension of the excellent results of Ma and Ge [18].

## 2. Main result

In order to abbreviate our discussion, we need the following observations and lemmas.

Throughout this paper, we assume that:

(i)  $q$  is a constant and satisfies  $\frac{1}{p} + \frac{1}{q} = 1$ ;

and observe that:

(ii)  $(\phi_p)^{-1}(z) := \phi_q(z) = |z|^{q-2}z$ .

**Definition 1.** Let  $X$  be a real Banach space and  $P$  be a cone of  $X$ . A map  $\psi : P \rightarrow [0, +\infty)$  is called a nonnegative continuous concave functional map if  $\psi$  is nonnegative, continuous and satisfies for all  $x, y \in P$  and  $t \in [0, 1]$ ,

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y).$$

**Definition 2.** Let  $X$  be a real Banach space and  $P$  be a cone of  $X$ . A map  $\beta : P \rightarrow [0, +\infty)$  is called a nonnegative continuous convex functional map if  $\beta$  is nonnegative, continuous and satisfies for all  $x, y \in P$  and  $t \in [0, 1]$ ,

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y).$$

Let  $\gamma, \beta$  and  $\theta$  be nonnegative continuous convex functionals on  $P$ , and let  $\alpha$  and  $\psi$  be nonnegative continuous concave functionals on  $P$ . For given nonnegative numbers  $h, a, b, d$  and  $c$ , we define the following sets:

$$\begin{aligned} P(\gamma, c) &= \{x \in P \mid \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in P \mid a \leq \alpha(x), \gamma(x) \leq c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in P \mid \beta(x) \leq d, \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P \mid a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}, \\ Q(\gamma, \beta, \psi, h, d, c) &= \{x \in P \mid h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}. \end{aligned}$$

In order to obtain multiple positive solutions of (BVP), the following fixed-point theorem due to Avery which is a generalization of the Leggett–Williams fixed-point theorem will be fundamental.

**Lemma 3** (Avery and Henderson [7], Theorem 2.4). Let  $X$  be a real Banach space and  $P$  be a cone of  $X$ . Suppose  $\gamma, \beta$  and  $\theta$  are three nonnegative continuous convex functionals on  $P$  and  $\alpha, \psi$  are two nonnegative continuous concave functionals on  $P$  such that there are  $c, L \in (0, \infty)$  satisfying

$$\alpha(x) \leq \beta(x), \quad \|x\| \leq L\gamma(x)$$

for  $x \in \overline{P(\gamma, c)}$ . Suppose further that

$$T : \overline{P(\gamma, c)} \rightarrow \overline{P(\gamma, c)}$$

is completely continuous and there exist  $h, d, a, b \geq 0$  with  $0 < d < a$  such that each of the following is satisfied:

- (i)  $\{x \in P(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$  and  $\alpha(Tx) > a$  for  $x \in P(\gamma, \theta, \alpha, a, b, c)$ ,
- (ii)  $\{x \in Q(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$  and  $\beta(Tx) < d$  for  $x \in Q(\gamma, \beta, \psi, h, d, c)$ ,
- (iii)  $\alpha(Tx) > a$  for  $x \in P(\gamma, \alpha, a, c)$  with  $\theta(Tx) > b$ ,
- (iv)  $\beta(Tx) < d$  for  $x \in Q(\gamma, \beta, d, c)$  with  $\psi(Tx) < h$ .

Then,  $T$  has at least three fixed points  $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$  such that  $\beta(x_1) < d, a < \alpha(x_2), d < \beta(x_3)$  with  $\alpha(x_3) < a$ .

**Lemma 4** (Ma and Ge [18]). Suppose that  $H$  is continuous on  $[0, 1]$ , then the unique solution of the boundary value problem:

$$\begin{cases} -y'' = H(t) & \text{in } (0, 1) \\ y(0) = \phi_p(\mu)y(\delta), \quad y(1) = \phi_p(\nu)y(\delta) \end{cases} \quad (1)$$

is

$$y(t) = \frac{1}{M} \int_0^1 g(t, s) H(s) ds,$$

where

$$M = 1 - \phi_p(\mu) - (\phi_p(v) - \phi_p(\mu))\delta \neq 0 \quad (2)$$

and

$$g(t, s) = \begin{cases} s(1-t) + \phi_p(v)s(t-\delta), & 0 \leq s \leq t < \delta < 1 \text{ or } 0 \leq s \leq \delta \leq t \leq 1, \\ t(1-s) + \phi_p(v)t(s-\delta) + \phi_p(\mu)(1-\delta)(s-t), & 0 \leq t \leq s \leq \delta < 1, \\ s(1-t) + \phi_p(v)\delta(t-s) + \phi_p(\mu)(1-t)(\delta-s), & 0 \leq \delta \leq s \leq t \leq 1, \\ (1-s)(t - \phi_p(\mu)t + \phi_p(\mu)\delta), & 0 < \delta \leq t \leq s \leq 1 \text{ or } 0 \leq t < \delta \leq s \leq 1. \end{cases}$$

**Lemma 5** (Ma and Ge [18]). Suppose that  $H$  is continuous on  $[0, 1]$ , then the unique solution of the boundary value problem:

$$\begin{cases} -y'' = H(t) & \text{in } (0, 1), \\ y(0) = \xi y(1), & y'(1) = \eta y'(0) \end{cases} \quad (3)$$

is

$$y(t) = \frac{1}{M_1} \int_0^1 h(t, s) H(s) ds,$$

where

$$M_1 = (1 - \xi)(1 - \eta) \neq 0$$

and

$$h(t, s) = \begin{cases} s + \eta(t-s) + \xi\eta(1-t), & 0 \leq s \leq t \leq 1, \\ t + \xi(s-t) + \xi\eta(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

**Lemma 6** (Ma and Ge [18]). Suppose that  $0 \leq \xi, \eta < 1$ ,  $0 < t_1 < t_2 < 1$  and  $\delta \in (0, 1)$ . Then, for all  $s \in [0, 1]$ ,

$$\frac{h(t_1, s)}{h(t_2, s)} \geq \frac{t_1}{t_2} \quad (4)$$

and

$$\frac{h(1, s)}{h(\delta, s)} \leq \frac{1}{\delta} \quad (5)$$

hold.

**Lemma 7** (Ma and Ge [18]). Suppose that  $\xi, \eta > 1$ ,  $0 < t_1 < t_2 < 1$  and  $\delta \in (0, 1)$ . Then, for all  $s \in [0, 1]$ ,

$$\frac{h(t_2, s)}{h(t_1, s)} \geq \frac{1 - t_2}{1 - t_1} \quad (6)$$

and

$$\frac{h(0, s)}{h(\delta, s)} \leq \frac{1}{1 - \delta} \quad (7)$$

hold.

### 3. Three positive solutions

Here, we consider the classical Banach space  $X = C^n([0, 1])$  which is equipped with the usual norm. The cones  $P_1$  and  $P_2 \subset X$  are defined as follows:

$$P_1 = \{u \in X \mid u^{(n-2)}(t) \text{ is nonnegative concave and nondecreasing on } (0, 1)\}$$

and

$$P_2 = \{u \in X \mid u^{(n-2)}(t) \text{ is nonnegative concave and nonincreasing on } (0, 1)\}.$$

Next, let  $t_1, t_2, t_3 \in (0, 1)$  with  $t_1 < t_2$  be fixed. Moreover, we shall define the nonnegative continuous concave functionals  $\alpha, \psi$  and nonnegative convex functionals  $\beta, \theta, \gamma$  on  $P_1$  by

$$\begin{aligned}\gamma(x) &= \max_{t \in [0, t_3]} x^{(n-2)}(t) = x^{(n-2)}(t_3), \quad x \in P_1, \\ \psi(x) &= \min_{t \in [\delta, 1]} x^{(n-2)}(t) = x^{(n-2)}(\delta), \quad x \in P_1, \\ \beta(x) &= \max_{t \in [\delta, 1]} x^{(n-2)}(t) = x^{(n-2)}(1), \quad x \in P_1, \\ \alpha(x) &= \min_{t \in [t_1, t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_1), \quad x \in P_1 \\ \theta(x) &= \max_{t \in [t_1, t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_2), \quad x \in P_1.\end{aligned}$$

**Theorem 8** (First Existence Theorem). Suppose that  $M > 0$  and the following assumptions hold:

(C<sub>1</sub>)  $f : (0, 1) \times \mathbb{R}^{n+2} \rightarrow [0, \infty)$  is continuous,

(C<sub>2</sub>)  $\xi, \eta \in [0, 1]$  and  $a, b, c \in (0, \infty)$  satisfy  $0 < a < b < \frac{t_2}{t_1}b \leq c$   
and

(C<sub>3</sub>) there are three positive constants  $k_1, k_2, k_3$  satisfying the following conditions:

$$\begin{aligned}(1^\circ) \quad & f(t, y_1, \dots, y_{n+2}) > k_1 \phi_p \left( \frac{b}{B} \right), \\ & \text{for } (t, y_1, \dots, y_{n+2}) \in [t_1, t_2] \times \prod_{k=0}^{n-2} \left[ bt_1^{n-2-k}, \frac{t_2^{n-1-k}}{t_1} b \right] \times \mathbb{R}^3, \\ (2^\circ) \quad & f(t, y_1, \dots, y_{n+2}) < k_2 \phi_p \left( \frac{a}{C} \right), \\ & \text{for } (t, y_1, \dots, y_{n+2}) \in [0, 1] \times [0, a]^{n-1} \times \mathbb{R}^3, \\ (3^\circ) \quad & f(t, y_1, \dots, y_{n+2}) \leq k_3 \phi_p \left( \frac{c}{A} \right), \\ & \text{for } (t, y_1, \dots, y_{n+2}) \in [0, 1] \times \left[ 0, \frac{1}{t_3} c \right]^{n-1} \times \mathbb{R}^3.\end{aligned}$$

where  $A, B$  and  $C$  are defined as follows:

$$\begin{aligned}A &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left[ \int_0^1 k_3 \cdot g(s, r) dr \right] ds, \\ B &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left[ \int_{t_1}^{t_2} k_1 \cdot g(s, r) dr \right] ds, \\ C &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left[ \int_0^1 k_2 \cdot g(s, r) dr \right] ds.\end{aligned}$$

Then, the boundary value problem (BVP) has at least three positive solutions.

**Proof.** By Lemma 4, we obtain

$$u^{(n)}(t) = -\frac{1}{\phi_q(M)} \phi_q \left( \int_0^1 g(t, s) f(s, u(s), \dots, u^{(n+1)}(s)) ds \right).$$

Moreover, it follows from Lemma 5 that

$$u^{(n-2)}(t) = \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t, s) \phi_q \left( \int_0^1 g(s, \tau) f(\tau, u(\tau), \dots, u^{(n+1)}(\tau)) d\tau \right) ds.$$

Inspired by the above-mentioned result, we can define a completely continuous operator  $T : P_1 \rightarrow X$  via

$$(Tu)^{(n-2)}(t) = \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds.$$

It is clear that  $u$  is a positive solution of (BVP) if and only if  $u$  is a fixed point of  $T$  on cone  $P_1$ . Since  $M$  and  $M_1 > 0$ ,

$$(Tu)^{(n-2)} \geq 0 \quad \text{for } u \in P_1.$$

Furthermore,

$$\begin{aligned} ((Tu)^{(n-2)}(t))' &= \frac{1-\xi}{M_1\phi_q(M)} \left\{ \eta \int_0^t \phi_q \left( \int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\dots,u^{(n+1)}(r))dr \right) ds \right. \\ &\quad \left. + \int_t^1 \phi_q \left( \int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\dots,u^{(n+1)}(r))dr \right) ds \right\} \\ &\geq 0, \end{aligned}$$

and

$$((Tu)^{(n-2)}(t))'' = -\frac{1}{\phi_q(M)} \phi_q \left( \int_0^1 g(t,r)f(r,u(r),u^{(1)}(r),\dots,u^{(n+1)}(r))dr \right) \leq 0,$$

which imply  $TP_1 \subset P_1$ . From  $u \in P_1$ , we see that

$$\begin{aligned} \gamma(u) &= u^{(n-2)}(t_3) = u^{(n-2)}(t_3 \times 1 + (1-t_3) \times 0) \\ &\geq t_3 u^{(n-2)}(1) + (1-t_3) u^{(n-2)}(0) \geq t_3 \times \max_{t \in [0,1]} |u^{(n-2)}(t)|. \end{aligned}$$

Therefore, for  $u \in \overline{P_1(\gamma, c)}$ , we have

$$0 \leq u^{(n-2)}(t) \leq \max_{t \in [0,1]} |u^{(n-2)}(t)| \leq \frac{1}{t_3} \gamma(u) \leq \frac{1}{t_3} c$$

on  $[0, 1]$  which implies

$$0 \leq u^{(k)}(t) \leq \frac{1}{t_3} c \quad \text{on } [0, 1], \quad k = 0, \dots, n-2.$$

This and  $(C_3)-(3^\circ)$  imply

$$\begin{aligned} \gamma((Tu)) &= \max_{t \in [0, t_3]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_3) \\ &= \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 g(s,r)f(r,u(r),u^{(1)}(r),\dots,u^{(n+1)}(r))dr \right) ds \\ &\leq \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 k_3 \cdot g(s,r) \phi_p \left( \frac{c}{A} \right) dr \right) ds \\ &= \frac{c}{A} \left\{ \frac{1}{M_1\phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 k_3 \cdot g(s,r) dr \right) ds \right\} = c. \end{aligned}$$

Therefore, we obtain that  $T : \overline{P_1(\gamma, c)} \rightarrow \overline{P_1(\gamma, c)}$ . Next, we separate the rest of the proof into the following steps:

**Step 1.** For some  $\varepsilon_1 \in (0, \frac{t_2}{t_1}b - b)$ , let

$$u_1^{(n-2)}(t) := b + \varepsilon_1;$$

then,

$$u_1 \in P_1 \left( \gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c \right) = \{x \in P_1 \mid b \leq \alpha(x), \theta(x) \leq \frac{t_2}{t_1}b, \gamma(x) \leq c\}.$$

This means that  $u_1 \in \{u \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c) \mid \alpha(u) > b\} \neq \emptyset$  is well-defined. Moreover, we have

$$u^{(n-2)}(t) \geq u^{(n-2)}(t_1) = \alpha(u) \geq b$$

and

$$u^{(n-2)}(t) \leq u^{(n-2)}(t_2) = \theta(u) \leq \frac{t_2}{t_1}b$$

on  $[t_1, t_2]$ , which imply, for  $u \in P_1(\gamma, \theta, \alpha, b, \frac{t_2}{t_1}b, c)$ ,

$$bt_1^{n-2-k} \leq u^{(k)}(t) \leq \frac{t_2^{n-1-k}}{t_1} b \quad \text{on } [t_1, t_2], \quad k = 0, \dots, n-2.$$

This and (C<sub>3</sub>)-(1°) imply

$$\begin{aligned}\alpha((Tu)) &= \min_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_1) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &> \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left( \int_{t_1}^{t_2} k_1 \cdot g(s, r) \cdot \phi_p \left( \frac{b}{B} \right) dr \right) ds \\ &= \frac{b}{B} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left( \int_{t_1}^{t_2} k_1 \cdot g(s, r) dr \right) ds \right\} = b.\end{aligned}$$

**Step 2.** For some  $\varepsilon_2 \in (0, a - \delta a)$ , let

$$u_2^{(n-2)}(t) := a - \varepsilon_2;$$

then,

$$u_2 \in Q_1(\gamma, \beta, \psi, \delta a, a, c) = \{x \in P_1 \mid \delta a \leq \psi(x), \beta(x) \leq a, \gamma(x) \leq c\}.$$

This means that  $u_2 \in \{u \in Q_1(\gamma, \beta, \psi, \delta a, a, c) \mid \beta(u) < a\} \neq \emptyset$  is well-defined. Moreover, we have  $0 \leq u^{(n-2)}(t) \leq u^{(n-2)}(1) = \beta(u) \leq a$  on  $[0, 1]$ , which implies, for  $u \in Q_1(\gamma, \beta, \psi, \delta a, a, c)$ ,

$$0 \leq u^{(k)}(t) \leq a \quad \text{on } [0, 1], \quad k = 0, \dots, n-2.$$

This and (C<sub>3</sub>)-(2°) imply

$$\begin{aligned}\beta((Tu)) &= \max_{t \in [\delta, 1]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(1) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &< \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left( \int_0^1 k_2 \cdot g(s, r) \cdot \phi_p \left( \frac{a}{C} \right) dr \right) ds \\ &= \frac{a}{C} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left( \int_0^1 k_2 \cdot g(s, r) dr \right) ds \right\} = a.\end{aligned}$$

**Step 3.** For  $u \in P_1(\gamma, \alpha, b, c) = \{x \in P_1 \mid b \leq \alpha(x), \gamma(x) \leq c\}$  with  $\theta(Tu) > \frac{t_2}{t_1} b$ , it follows from Lemma 6 that

$$\begin{aligned}\alpha((Tu)) &= \min_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_1) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_1, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 \frac{h(t_1, s)}{h(t_2, s)} h(t_2, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &\geq \frac{t_1}{t_2} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \right\} \\ &= \frac{t_1}{t_2} \{(Tu)^{(n-2)}(t_2)\} = \frac{t_1}{t_2} \max_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) \\ &= \frac{t_1}{t_2} \theta(Tu) > \frac{t_1}{t_2} \frac{t_2}{t_1} b = b.\end{aligned}$$

**Step 4.** For  $u \in Q_1(\gamma, \beta, a, c) = \{x \in P_1 \mid \beta(x) \leq a, \gamma(x) \leq c\}$  with  $\psi(Tu) < \delta a$ , it follows from Lemma 6 that

$$\begin{aligned}\beta((Tu)) &= \max_{t \in [\delta, 1]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(1) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(1, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 \frac{h(1, s)}{h(\delta, s)} h(\delta, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\delta} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(\delta, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \right\} \\
&= \frac{1}{\delta} \{(Tu)^{(n-2)}(\delta)\} = \frac{1}{\delta} \min_{t \in [\delta, 1]} (Tu)^{(n-2)}(t) \\
&= \frac{1}{\delta} \psi(Tu) < \frac{1}{\delta} \delta a = a.
\end{aligned}$$

Therefore, the hypotheses of Lemma 3 are satisfied. Thus, there exist three positive solutions  $u_1, u_2, u_3$  for (BVP).  $\square$

Let  $t_1, t_2, t_3 \in (0, 1)$  with  $t_1 < t_2$  fixed. Moreover, we shall define the nonnegative continuous concave functionals  $\alpha, \psi$  and nonnegative convex functionals  $\beta, \theta, \gamma$  on  $P_2$  by

$$\begin{aligned}
\gamma(x) &= \max_{t \in [t_3, 1]} x^{(n-2)}(t) = x^{(n-2)}(t_3), \quad x \in P_2, \\
\psi(x) &= \min_{t \in [0, \delta]} x^{(n-2)}(t) = x^{(n-2)}(\delta), \quad x \in P_2, \\
\beta(x) &= \max_{t \in [0, \delta]} x^{(n-2)}(t) = x^{(n-2)}(0), \quad x \in P_2, \\
\alpha(x) &= \min_{t \in [t_1, t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_2), \quad x \in P_2, \\
\theta(x) &= \max_{t \in [t_1, t_2]} x^{(n-2)}(t) = x^{(n-2)}(t_1), \quad x \in P_2.
\end{aligned}$$

**Theorem 9** (Second Existence Theorem). Suppose that  $M > 0$  and the following assumptions hold:

(C<sub>4</sub>)  $f : (0, 1) \times \mathbb{R}^{n+2} \rightarrow [0, \infty)$  is continuous,

(C<sub>5</sub>)  $\xi, \eta \in (1, \infty)$  and  $a, b, c \in (0, \infty)$  satisfy  $0 < a < b < \frac{1-t_1}{1-t_2} b \leq c$ ,  
and

(C<sub>6</sub>) there are three positive constants  $k_1, k_2, k_3$  satisfying (2°), (3°), and

$$\begin{aligned}
(4^\circ) \quad &f(t, y_1, \dots, y_{n+2}) > k_1 \phi_p \left( \frac{b}{B} \right), \\
&(t, y_1, \dots, y_{n+2}) \in [t_1, t_2] \times \prod_{k=0}^{n-2} [bt_1^{n-2-k}, \frac{1-t_1}{1-t_2} t_2^{n-2-k} b] \times \mathbb{R}^3.
\end{aligned}$$

Then, the boundary value problem (BVP) has at least three positive solutions.

**Proof.** By Lemma 4, we obtain

$$u^{(n)}(t) = -\frac{1}{\phi_q(M)} \phi_q \left( \int_0^1 g(t, s) f(s, u(s), \dots, u^{(n+1)}(s)) ds \right).$$

Moreover, it follows from Lemma 5 that

$$u^{(n-2)}(t) = \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t, s) \phi_q \left( \int_0^1 g(s, \tau) f(\tau, u(\tau), \dots, u^{(n+1)}(\tau)) d\tau \right) ds.$$

Inspired by the above-mentioned result, we can define a completely continuous operator  $T : P_2 \rightarrow X$  via

$$(Tu)^{(n-2)}(t) = \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds.$$

It is clear that  $u$  is a positive solution of (BVP) if and only if  $u$  is a fixed point of  $T$  on cone  $P_2$ . Since  $M$  and  $M_1 > 0$ ,

$$(Tu)^{(n-2)} \geq 0 \quad \text{for } u \in P_2.$$

Furthermore,

$$\begin{aligned}
((Tu)^{(n-2)}(t))' &= \frac{1-\xi}{M_1 \phi_q(M)} \left\{ \eta \int_0^t \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \right. \\
&\quad \left. + \int_t^1 \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \right\} \\
&\leq 0,
\end{aligned}$$

and

$$((Tu)^{(n-2)}(t))'' = -\frac{1}{\phi_q(M)}\phi_q\left(\int_0^1 g(t,r)f(r,u(r),u^{(1)}(r),\dots,u^{(n+1)}(r))dr\right) \leq 0,$$

which imply  $TP_2 \subset P_2$ . From  $u \in P_2$ , we see that

$$\begin{aligned}\gamma(u) &= u^{(n-2)}(t_3) = u^{(n-2)}(t_3 \times 1 + (1-t_3) \times 0) \\ &\geq t_3 u^{(n-2)}(1) + (1-t_3)u^{(n-2)}(0) \geq t_3 \times \max_{t \in [0,1]} |u^{(n-2)}(t)|.\end{aligned}$$

Therefore, we have

$$0 \leq u^{(n-2)}(t) \leq \max_{t \in [0,1]} |u^{(n-2)}(t)| \leq \frac{1}{t_3} \gamma(u) \leq \frac{1}{t_3} c$$

on  $[0, 1]$ , which implies, for  $u \in \overline{P_2(\gamma, c)}$ ,

$$0 \leq u^{(k)}(t) \leq \frac{1}{t_3} c \quad \text{on } [0, 1], k = 0, \dots, n-2.$$

This and  $(C_6)$ -(3°) imply

$$\begin{aligned}\gamma((Tu)) &= \max_{t \in [t_3, 1]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_3) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &\leq \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 k_3 \cdot g(s, r) \phi_p \left( \frac{c}{A} \right) dr \right) ds \\ &= \frac{c}{A} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_3, s) \phi_q \left( \int_0^1 k_3 \cdot g(s, r) dr \right) ds \right\} = c.\end{aligned}$$

Therefore, we obtain that  $T : \overline{P_2(\gamma, c)} \rightarrow \overline{P_2(\gamma, c)}$ . Next, we separate the rest of the proof into the following steps:

**Step 1.** For some  $\varepsilon_1 \in (0, \frac{1-t_1}{1-t_2}b - b)$ , let

$$u_1^{(n-2)}(t) := b + \varepsilon_1;$$

then,

$$u_1 \in P_2 \left( \gamma, \theta, \alpha, b, \frac{1-t_1}{1-t_2}b, c \right) = \left\{ x \in P_2 \mid b \leq \alpha(x), \theta(x) \leq \frac{1-t_1}{1-t_2}b, \gamma(x) \leq c \right\}.$$

This means that  $u_1 \in \{u \in P_2(\gamma, \theta, \alpha, b, \frac{1-t_1}{1-t_2}b, c) \mid \alpha(u) > b\} \neq \emptyset$  is well-defined. Moreover, we have

$$u^{(n-2)}(t) \geq u^{(n-2)}(t_2) = \alpha(u) \geq b$$

and

$$u^{(n-2)}(t) \leq u^{(n-2)}(t_1) = \theta(u) \leq \frac{1-t_1}{1-t_2}b$$

on  $[t_1, t_2]$ , which imply, for  $u \in P_2(\gamma, \theta, \alpha, b, \frac{1-t_1}{1-t_2}b, c)$ ,

$$b_1^{n-2-k} \leq u^{(k)}(t) \leq \frac{1-t_1}{1-t_2} t_2^{n-2-k} b \quad \text{on } [t_1, t_2], k = 0, \dots, n-2.$$

This and  $(C_6)$ -(4°) imply

$$\begin{aligned}\alpha((Tu)) &= \min_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_2) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), u^{(1)}(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &> \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_{t_1}^{t_2} k_1 \cdot g(s, r) \cdot \phi_p \left( \frac{b}{B} \right) dr \right) ds \\ &= \frac{b}{B} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_{t_1}^{t_2} k_1 \cdot g(s, r) dr \right) ds \right\} = b.\end{aligned}$$



**Step 2.** For some  $\varepsilon_2 \in (0, a - (1 - \delta)a)$ , let

$$u_2^{(n-2)}(t) := a - \varepsilon_2;$$

then

$$u_2 \in P_2(\gamma, \beta, \psi, (1 - \delta)a, a, c) = \{x \in P_2 \mid (1 - \delta)a \leq \psi(x), \beta(x) \leq a, \gamma(x) \leq c\}.$$

This means that  $u_2 \in \{u \in Q_2(\gamma, \beta, \psi, (1 - \delta)a, a, c) \mid \beta(u) < a\} \neq \emptyset$  is well-defined. Moreover, we have

$$0 \leq u^{(n-2)}(t) \leq u^{(n-2)}(0) = \beta(u) \leq a$$

on  $[0, 1]$ , which implies, for  $u \in Q_2(\gamma, \beta, \psi, (1 - \delta)a, a, c)$ ,

$$0 \leq u^{(k)}(t) \leq a \quad \text{on } [0, 1], \quad k = 0, \dots, n - 2.$$

This and  $(C_6)$ -(2°) imply

$$\begin{aligned} \beta((Tu)) &= \max_{t \in [0, \delta]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(0) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(0, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &< \frac{1}{M_1 \phi_q(M)} \int_0^1 h(0, s) \phi_q \left( \int_0^1 k_2 \cdot g(s, r) \cdot \phi_p \left( \frac{a}{C} \right) dr \right) ds \\ &= \frac{a}{C} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(0, s) \phi_q \left( \int_0^1 k_2 \cdot g(s, r) dr \right) ds \right\} = a. \end{aligned}$$

**Step 3.** For  $u \in P_2(\gamma, \alpha, b, c) = \{x \in P_2 \mid b \leq \alpha(x), \gamma(x) \leq c\}$  with  $\theta(Tu) > \frac{1-t_1}{1-t_2}b$ , it follows from Lemma 7 that

$$\begin{aligned} \alpha(Tu) &= \min_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(t_2) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 \frac{h(t_2, s)}{h(t_1, s)} h(t_1, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &\geq \frac{1-t_2}{1-t_1} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(t_2, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \right\} \\ &= \frac{1-t_2}{1-t_1} \{(Tu)^{(n-2)}(t_1)\} = \frac{1-t_2}{1-t_1} \max_{t \in [t_1, t_2]} (Tu)^{(n-2)}(t) \\ &= \frac{1-t_2}{1-t_1} \theta(Tu) > \frac{1-t_2}{1-t_1} \times \frac{1-t_1}{1-t_2} b = b. \end{aligned}$$

**Step 4.** For  $Q_2(\gamma, \beta, a, c) = \{x \in P_2 \mid \beta(x) \leq a, \gamma(x) \leq c\}$  with  $\psi(Tu) < (1 - \delta)a$ , it follows from Lemma 7 that

$$\begin{aligned} \beta(Tu) &= \max_{t \in [0, \delta]} (Tu)^{(n-2)}(t) = (Tu)^{(n-2)}(0) \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 h(0, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &= \frac{1}{M_1 \phi_q(M)} \int_0^1 \frac{h(0, s)}{h(\delta, s)} h(\delta, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \\ &\leq \frac{1}{(1-\delta)} \left\{ \frac{1}{M_1 \phi_q(M)} \int_0^1 h(\delta, s) \phi_q \left( \int_0^1 g(s, r) f(r, u(r), \dots, u^{(n+1)}(r)) dr \right) ds \right\} \\ &= \frac{1}{(1-\delta)} \{(Tu)^{(n-2)}(\delta)\} = \frac{1}{(1-\delta)} \min_{t \in [0, \delta]} (Tu)^{(n-2)}(t) \\ &= \frac{1}{(1-\delta)} \psi(Tu) < \frac{1}{(1-\delta)} (1 - \delta)a = a. \end{aligned}$$

Therefore, the hypotheses of Lemma 4.2.3 are satisfied and there exist three positive solutions  $u_1, u_2, u_3$  for (BVP).  $\square$

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